Functional Programming
Lecture 3

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Higher-order functions


WHAT IF I TOLD YOU
YOU CAN PASS A
FUNCTION TO FUNCTION
ELLORA
SOCIAL

Higher-order functions

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- Allow capturing and reusing common programming patterns
- Provide higher level of abstraction
- The reason why functional programs are compact

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```
(apply append '((1 2 3) (a b))) => (1 2 3 a b)
```

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- **foldl** — aggregate elements in a list using a binary operation from left to right

Map

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(map fn '(a1 a2 ...)
        '(b1 b2 ...)
        '(c1 c2 ...))
     => ((fn a1 b1 c1)  (fn a2 b2 c2) ...)
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(map list-ref '((1 2 3)
                (4 5 6)
                (7 8 9)) (range 0 3)) => (1 5 9)
```

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$$foldl \quad b_0 \xrightarrow{f(a_1, b_0)} b_1 \xrightarrow{f(a_2, b_1)} b_2 \xrightarrow{f(a_3, b_2)} b_3 \xrightarrow{f(a_4, b_3)} b_4$$

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Folding

```
(define (trace mat)
  (foldl + 0 (map list-ref
                  mat
                  (range 0 (length mat)))))

(trace '((1 2 3) (4 5 6) (7 8 9))) => 15
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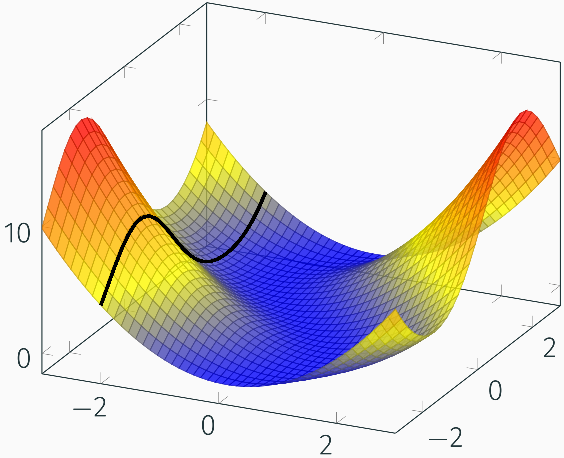
(foldl cons '() '(a b c)) => (c b a)
(foldr cons '() '(a b c)) => (a b c)
(foldl min +inf.0 '(0 -3 2)) => -3.0
```

Currying and compositions

Partial evaluation — Currying



Partial evaluation — Currying

$$\begin{aligned} f : A \times B \to C &\quad\Longrightarrow\quad \hat{f} : A \to (B \to C) \\ f : A \times B \times C \to D &\quad\Longrightarrow\quad \hat{f} : A \to (B \to (C \to D)) \end{aligned}$$

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(define (curried-list-ref lst)
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```
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```

Simplified syntax:

```
((curry list-ref '(a b c)) 2) => c
(map (curry * 2) '(1 2 3)) => (2 4 6)
```

Function composition

Let $f\colon A \to B$ and $g\colon B \to C$ be functions.

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A \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & B \arrow[r, "g"'] & C
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Compose function

Traditional style:

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Point-free style:

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(define str->alpha2
  (compose list->string
           (curry filter char-alphabetic?)
           string->list))
```

Example — morphic sequences

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Define sequence:

$$\begin{aligned}
\phi^0(0) &= 0 \\
\phi^1(0) &= h(0) = 011 \\
\phi^2(0) &= h(011) = 01100 \\
\phi^3(0) &= h(01100) = 01100011011 \\
&\vdots
\end{aligned}$$

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Binding scopes

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Scheme/Racket uses the lexical scope as most of the modern programming languages.

Function closures

```
(define (make-adder x) (lambda (y) (+ x y)))

(define adder1 (make-adder 10))
(define adder2 (make-adder 3))
(define adder3 (make-adder -7))
```

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adder1 -> (lambda (y) (+ 10 y))
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```
adder1 -> (x 10) (lambda (y) (+ x y))
adder2 -> (x 3) (lambda (y) (+ x y))
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A **function call** expression:

- Evaluates the code of a function closure
- In the environment of the function closure extended by with bindings for the arguments

Applications of closures

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(define (get-y p)
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(define p (point 3 10))
(get-x p) => 3
(get-y p) => 10
```

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(define pers (person "John"
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```

Accessor functions to all fields are automatically defined.

```
(person-first-name pers) => "John"
(person-surname pers) => "Down"
(person-age pers) => 33
(person? pers) => #t
(person? "John") => #f
```

Homework assignment

Homework assignment 1 — ASCII art generator

Aim: Practice applications of higher-order functions

```
;;;::::,,,,...
o;;;;::::,,,....
ooo;;;;:::,,,,....
xoooo;;;::::,,,....
xxoooo;;;;:::,,,,...
xxxxoooo;;;::::,,,,.
%%xxxxooo;;;;::::,,,
%%%%xxxoooo;;;;:::,,
##%%%xxxxoooo;;;::::
###%%%xxxoooo;;;;::
@####%%%xxxxoooo;;;:
@@@###%%%%xxxxooo;;;
```

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`ascii-art`

```
;;;::::,,,,...
o;;;;::::,,,....
ooo;;;;:::,,,,....
xoooo;;;::::,,,....
xxoooo;;;;:::,,,,...
xxxxoooo;;;::::,,,,.
%%xxxxooo;;;;::::,,,
%%%%xxxoooo;;;;:::,,
##%%%xxxxoooo;;;::::
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```

	.	,	:	;	o	x	\%	\#	@

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Description: all details can be found in CW

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- Scheme/Racket uses lexical scope.
- Function closures are pairs storing the code of a function together with the current environment.